Envelope Equations

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The envelope is defined as a curve which touches the members of a family of curves each at one point and is tangent to the member at that point.An example of a family of curves- in this case, all of them are straight lines in the x,y plane- is the following:

$$\frac{x}{a} + \frac{y}{b} = c \tag{1}$$

Equation (1) gives a line for every value of a and b. Thus,it represents a family of curves. The envelope curve, by definition, touches each of the curve but only once. Lets develop general equations that must be fulfilled by the envelope. Let us define a general family of curves with three variables.

$$f(x, y, \gamma) = 0 \tag{2}$$

The envelope satisfies equation (2) as it touches each and every member of the family of curves.

To find another equation that the envelope must satisfy we do the following.Let us differentiate equation (2) with respect to γ.γ is independent and x and y are dependent on γ.

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial \gamma} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \gamma} + \frac{\partial f}{\partial \gamma} = 0 \tag{3}$$

Since the envelope curve is tangent to the member at the point of contact, we can incorporate another property-that the slope of the memeber and the envelope is the same (at point of contact) into an equation that defines the envelope.
To do so,let us differentiate equation (2) with respect to x.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0 \tag{4}$$

The above equation does not contain the derivative of γ with resepect to x as γ does not depend on x(i.e. $\frac{\partial \gamma}{\partial x} = 0$).The above equation can be arranged into the following:

$$\frac{\partial y}{\partial x} = -\frac{f_x}{f_y} \tag{5}$$

where f_x represents $\frac{\partial f}{\partial x}$. Equation (5) gives the slope of the family of curves.

$$\frac{\partial y}{\partial x} = lim_{\gamma \to 0} \frac{\frac{\partial y}{\partial \gamma}}{\frac{\partial x}{\partial \gamma}} \tag{6}$$

Equation (6) represents the slope of the envelope. Since the slope of the envelope is tangent to the family of curves; equating (5) and (6) , we get:

$$-f_x \frac{\partial x}{\partial \gamma} = f_y \frac{\partial y}{\partial \gamma} \tag{7}$$

Using equation (7) in equation (3),we get

$$\frac{\partial f}{\partial \gamma} = 0 \tag{8}$$

To sum up, the envelope of a family of curves satisfies the following equations:

$$\frac{\partial f}{\partial \gamma} = 0$$

$$f(x, y, \gamma) = 0$$

From the above two equations, the variable γ can be eliminated and thus the envelope can be represented in the x-y plane.